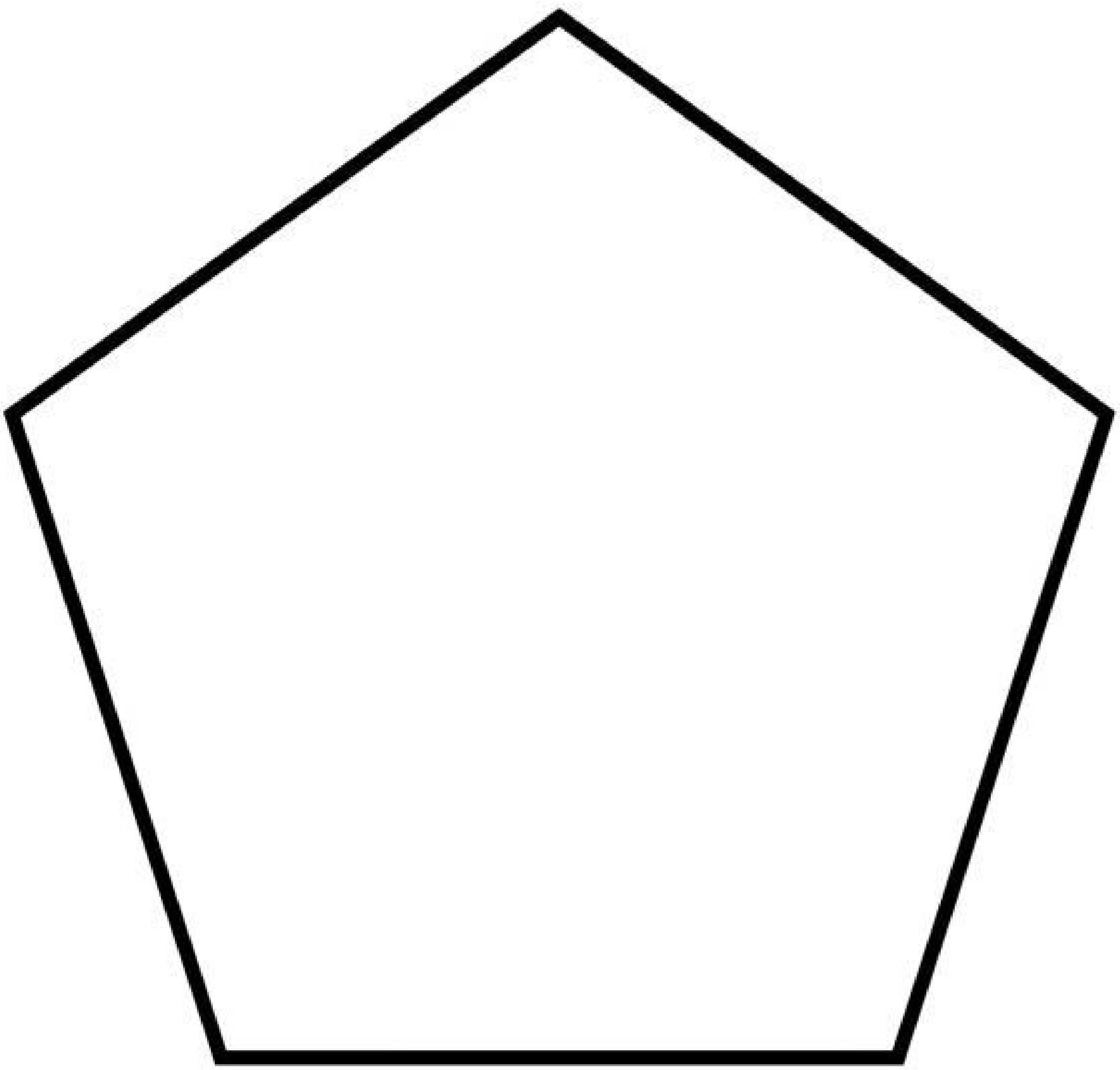
Question:

What is the area of a regular pentagon with side length 1? Please express answer without using trigonometry functions.



See next page for the answer.

Answer:

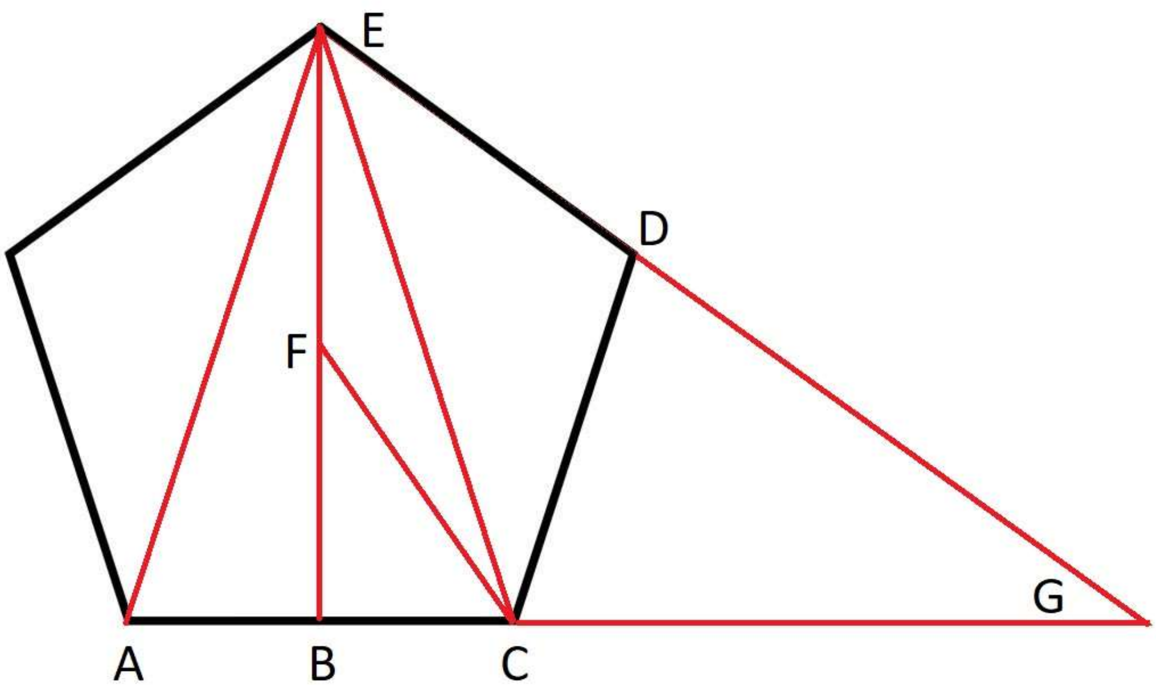
There are various ways of expressing the answer. I prefer…

$$\frac{\sqrt{5\times(5+2\sqrt{5})}}{4}$$ = ~ 1.72047740058897.

Scroll down for a solution.

Solution

First, let's label some points to solve for the angles.



We know a circle has 360 degrees. We would divide that by 5 to get the interior angle of each pentagon. So ∢ AFB = 360°/5 = 72°. Dividing that by 2 gives us ∢ BFC = 72°/2 = 36°.

∢BFC = 90, so ∢ BCF = 180 – 90 – 36 = 54.

Since FC bisects ∢ BCD, we can double ∢BCF to get ∢ BCD = 108.

∢CDG = 180 - ∢EDC = 180 – 108 = 72.

∢ DCG = ∢ CDG = 72.

Thus, ∢ CGD = 180 – ∢ CDG - ∢ DCG = 180 – 72 – 72 = 36.

Next, let's look at isosceles triangle EDC. We know ∢ EDC = 108.

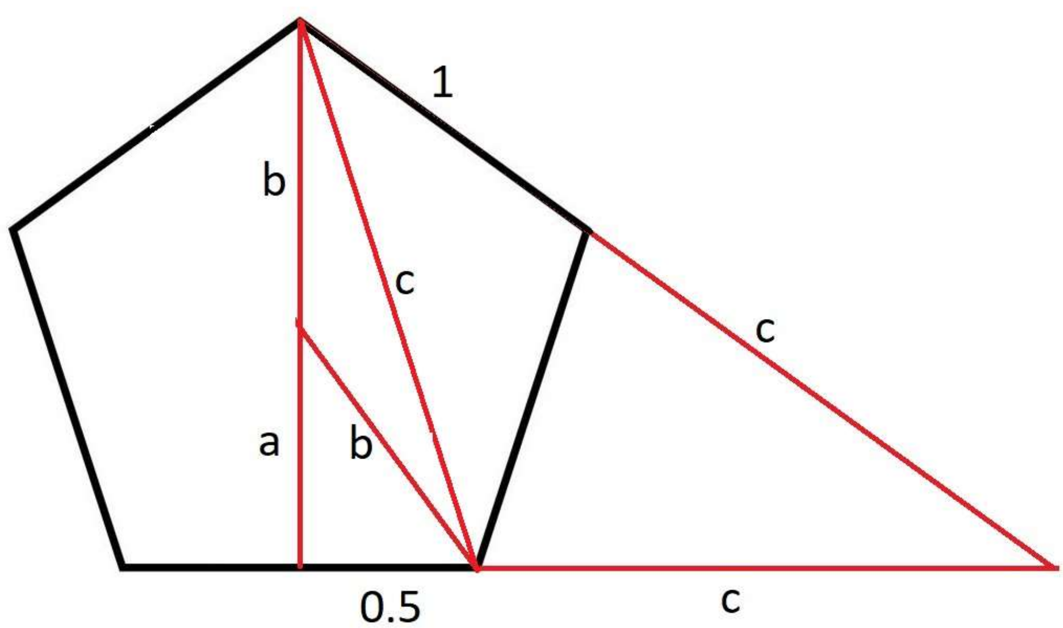
Thus, ∢ CED and ∢ ECD each equal = (180-108)/2 = 36.

We can now solve for ∢ AEC = 108 – 2*∢ CED = 180 – 2*36 = 36.

Notice that ∢ CGD = ∢ AEC.

Look carefully and you'll see that triangles ACE and CDG are equal! Each has one side of length 1 and two sides of length AE=CE=CG=DG. This is key to solving the whole problem!

Next, let's switch to another diagram to work on the distances between points. Please don't get confused with the change in notation. I use lower case for distances and upper case for angles. Remember, we just showed in the previous diagram that EC = CG = DG.



Let's use the Pythagorean formula to solve for a, b, and c.

(1) $a^2 + ¼ = b^2$

(2) $(a+b)^2 + ¼ = c^2$

(3) $(c + ½)^2 + (a+b)^2 = (c+1)^2$

Let's substitute $c^2 - ¼$ for $(a+b)^2$ in equation (3):

$(c + ¼)^2 + c^2 - ¼ = (c+1)^2$

This reduces to c2 – c – 1 = 0

Using the Quadratic equation gives us c = $\frac{(1+\sqrt{5})}{2}$ =~ 1.618033989, which is the Golden Ratio!

Next, let's go back to equation (2), substituting this value for c.

$(a+b)^2 = (1+\sqrt{5})^2/4 - 1/4$

A few steps of basic algebra gives us:

$a + b = \frac{\sqrt{5+2\sqrt{5}}}{2}$ =~ 1.538841769

Next, let's find the area of triangle BEG from the first diagram.

We know the heigh is a+b and the base is c + ½.

Since it's a right triangle, the area is:

$$(1/2) \times (1/2) \times \frac{\sqrt{5+2\sqrt{5}}}{2} \times \left(\frac{(1+\sqrt{5})}{2} + \frac{1}{2}\right)$$

This can be simplified to:

$$\frac{\sqrt{5+2\sqrt{5}} \times (1+\sqrt{5})}{8} =\sim 1.629659585$$

Next, let's refer to the first diagram and find the area of triangle ACE.

We know the height is a+b and the base is 1. Thus the area of ACE is:

ACE = $\frac{\sqrt{5+2\sqrt{5}}}{4}$ =~ 0.769420884

As already established, triangle CDG = triangle AEC = $\frac{\sqrt{5+2\sqrt{5}}}{4}$.

The area of the pentagon can be expressed as 2× (BEG – CDG).

That works out to:

$$2 \times [\frac{\sqrt{5+2\sqrt{5}} \times (1+\sqrt{5})}{8} - \frac{\sqrt{5+2\sqrt{5}}}{4}]$$

After some tedious algebra, we can reduce this to:

$\frac{\sqrt{5\times(5+2\sqrt{5})}}{4}$ = ~ 1.72047740058897.